

The formation of share market prices under heterogeneous beliefs and common knowledge

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May 17, 2011

Abstract

Financial economic models often assume that investors know (or agree on) the fundamental value of the shares of the firm, easing the passage from the individual to the collective dimension of the financial system generated by the Share Exchange over time. Our model relaxes that heroic assumption of one unique "true value" and deals with the formation of share market prices through the dynamic formation of individual and social opinions (or beliefs) based upon a fundamental signal of economic performance and position of the firm, the forecast revision by heterogeneous individual investors, and their social mood or sentiment about the ongoing state of the market pricing process. Market clearing price formation is then featured by individual and group dynamics that make its collective dimension irreducible to its individual level. This dynamic holistic approach can be applied to better understand the market exuberance generated by the Share Exchange over time.

PACS numbers: Financial markets, 89.65.Gh; Social systems, 89.65.-s; Economics, 89.65.Gh

Keywords: opinion dynamics, reaction-diffusion, bubble formation, market dynamics

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1 Introduction

Advances in heterogeneous agents modeling from economics [1, 2] and complex systems dynamics in sociophysics [3, 4] call for an understanding of the working of the financial market based upon the collective and dynamic properties of systems featured by interacting parts and structures. These elements can be atoms or macromolecules in a physical context, as well as people, firms or regulated Exchanges in a socio-economic context. These approaches aim then to analyze the properties of socio-economic systems over time by focusing on interactions, relationships and the overall architecture of them.

Drawing upon these advances, this paper integrates the phenomenon of opinion dynamics studied by sociophysics [5, 6] to an economic dynamic model of market price formation over time through hazard and interaction [7]. The study of opinion dynamics has been a long and intensive subject of research among physicists working in sociophysics [8, 9, 10, 11, 12, 13, 14]: we apply here the Galam sequential probabilistic majority model of opinion dynamics [14, 15, 16].

During the last decades, financial market analysis has assisted to the proliferation of financial economic models that relax received assumptions of full knowledge, individual rationality and market efficiency. However, many models remain somewhat tied to an equilibrium approach to the formation of share market prices over time. This approach entails a pricing rule that satisfies all the market orders simultaneously passed by all investors in the purpose to maximize their expected utilities. This approach actually implies a peculiar understanding of the market coordination between individual investors. This coordination is supposed to be achieved in a solitary moment beyond time and context [17] when all investors contemplate the past, present and future of the business firm and univocally agree on its fundamental value of reference.

Once this unanimous consensus achieved, they perform market transactions at that price, which does not change unless the fundamental value of the firm does change [18]. Therefore, the share market price is supposed to incorporate (all the available information on) the fundamental value of the firm at every instant [19, 20]. The share market price becomes a sufficient statistics of the fundamental value of the firm [21], and investors are then supposed to know (or agree with) the fundamental value of its shares, even though the current market price may diverge from this "true value" in some ways over time. The understanding and the modeling of market pricing, and the dynamics of individual and collective opinions, are then driven by this assumption of uniqueness of the value of the firm.

Our model relaxes this heroic assumption of the market price as the best evidence of the "true value," and deals with the formation of share market price of one firm through the dynamic formation of individual and social opinions (or beliefs) based upon a fundamental signal F_t on the economic performance and position of the firm, the market clearing price of each share p_t , and a social mood (or sentiment) m_t on the ongoing state of market pricing process. Accordingly, individual investors are assumed to form their personal opinions - which orient their financial decisions of sell or hold, and buy or wait - in a

fundamentally interactive context [22]. At every instant k , each investor i does form its opinions respectively on the evolution of corporate fundamentals and the market clearing price that is continuously changed by achieved transactions through the Share Exchange.

Nothing can assure one investor about the permanent alignment between his opinion on the evolving fundamentals, its opinion on the current market price, and the market price itself [23]; nor can he be sure that the market order - which he passes through the Share Exchange according to those opinions - may be eventually satisfied. In this dynamic setting, the formation of share prices critically depends on both the interactive formation of social opinions among investors, and their common knowledge of corporate fundamentals over time. Every investor strives then to revise its price expectations $E_t(p_{t+1})|_i$ according to the dynamics of the fundamental signal F_t and the social market sentiment m_t .

2 Definition of Variables and Timing

The formation of share market price over time depends here on the dynamic formation of individual and social opinions (or beliefs) based upon a fundamental signal F_t on the economic performance and position of the firm, the market clearing price of each share p_t , and a social mood (or sentiment) m_t^j about the market pricing. These three dimensions (or layers, or orders) correspond to three different rhythms of change, that is, three different timings:

- $F_{t,h}$, the fundamental signal, has the slowest rhythm or the largest lag (duration). This means that F_t can be constant for $t + h$ periods; it lasts for h periods;
- p_t , the market clearing price, when exist, changes at each period t ;
- $m_{t,k}^j$, the social mood, has the quickest rhythm or the shortest lag. At each period t , its value is the final result of k interactions; each mood lasts indeed for $\frac{1}{k}$ periods.

Two distinctive forces drive then the market clearing price formation trough time. From one side, ongoing market pricing is submitted to individual guesses and intentions, hopes and fears, subsumed by the social mood $m_{t,k}^j$ and its quickest interactions; from another side, it is concerned with the slowest history of reporting and disclosure that, in principle, may be partly public, consistent, and conventionally agreed. This general *system* (which is no longer an equilibrium)¹ consists in and depends upon the coherence and universal diffusion of relevant and reliable knowledge through a price system (providing market information) and an accounting system (disclosing firm-specific, fundamental information) publicly determined and announced.

In particular, the **fundamental signal** is assumed to be common knowledge among all investors:

¹Our analysis distinguishes system and equilibrium as distinctive concepts.

- F_t is the fundamental signal about the economic performance and position generated by the business firm over time; it is fundamentally related to the firm's share price, but agents do not know (or agree on) the working of this relationship;
- F_t can be positive or negative and is exogenous to the model;
- Each agent applies an individual weight $\varphi_i \in [0; 1]$ to this signal, related to its personal confidence degree on it, from $\varphi_i = 0$ (no confidence at all) to $\varphi_i = 1$ (full confidence); this implies that all agents agree on the direction (sign) of the fundamental signal, but disagree on its material impact on the share price.
- In some specifications of the model, F_t may influence the social mood $m_{t,k}^j$.

The **social mood** (or market sentiment) captures the group interaction that generates the collective opinion on the current state of market pricing:

- $m_{t,k}^j \in [0; 1]$ is the mood of group j at time t , resulting from k group interactions (steps) starting from $m_{t,k=0}^j$;
- At each time t , $m_{t,k=0}^j \in [0; 1]$ exists and is exogenous to the model; in fact, $m_{t,k=0}^j$ may be endogenous to the model; in particular, it may depends on F .

The **market clearing price** (when exists) is generated by the matching of aggregate supply and demand, which are based upon heterogeneous price expectations by individual agents:

- p_t is the market clearing price at time t ;
- By assumption, $p_t \geq 0$;
- $E_t(p_{t+1})|_i^j$ is the price expectation at time t by agent i belonging to the group j on the market clearing price at period $t + 1$.

Individual investors have both group and individual heterogeneities regarding the formation of their expectations, which are then based upon individual and social opinions (or beliefs). In particular:

- Investors are distinguished between actual and potential shareholders. Analytically, they belong then to two groups $j = S, D$, where S denotes supply by potential sellers (actual shareholders), while D denotes demand by potential buyers (potential shareholders);
- In each group j , the number of agents is normalized to one, with $i \in [0; 1]$;
- In each group j , every agent i is characterized by an individual weight $\varphi_i \in [0; 1]$ that is applied to the fundamental signal F_t ;

- In each group j , agents are further characterized by the social mood $m_{t,k}^j$ that constitutes the market sentiment expressed by group j at time t ; its weight results from k inter-individual interactions between $t - 1$ and t .

3 The formation of individual expectations

Following [24] and [25], every agent forms its price expectation according to the following generic function:

$$E_t(p_{t+1})|_i^j = p_t + m_{t,k}^j (p_t - p_{t-1}) - \beta_i^j \left(E_{t-1}(p_t)|_i^j - p_t \right) + \gamma^j \varphi_i F_t \quad (1)$$

with $j = S$ (Supply), D (Demand); $i, \varphi_i \in [0, 1]$; $m_{t,k}^j \in [0, 1]$; $\beta_i^j \in [0, 1]$; $\gamma^j > 0$, and

$$\varepsilon_{i,t}^j \equiv \left(E_{t-1}(p_t)|_i^j - p_t \right).$$

This equation comprises four elements. The first element is the past clearing price p_t . The second element is the market signal (or price trend) that is weighted by the social opinion $m_{t,k}^j$ of the group j at time t , expressing the group's ongoing market confidence. The third element is the individual forecasting revision that consists of the difference between the past price expectation and the current realized price. This revision is weighted by β_i^j which may include both group and individual heterogeneities. The fourth element denotes the formation of an individual opinion by investor i (belonging to the group j) based upon available fundamental information F_t , which is common knowledge for both groups and all the individual investors, and is weighted then by the individual parameter φ_i . This structure of individual expectations follows the dual structure which the share market process is embedded in: From the cognitive viewpoint, investors are confronted with fundamental information from the business firm (they invest in) from one side, and the market pricing from another side. From the financial viewpoint, they are confronted with dividends and earnings generated by the business firm, and the capital gains and losses involved in the market trading (see [26] for further details). The firm side is subsumed here by the factor F , while the market side is captured by the price trend $_{t-1}\Delta_t(p)$.

Following the Galam's specification of the formation of social opinions [15, 16], we can define the generic function of the social mood $m_{t,k}^j$ as follows:

$$m_{t,k}^j = f \left(m_{t,k=0}^j, k_t^j, F_t, F_{t-1} \right)$$

where the fundamental signal F can influence k_t^j . For each group $j = S, D$, $m_{t,k}^j$ defines then the density at time t of individual investors who are confident in the market signal or trend ($m \rightarrow 1$), while $1 - m_{t,k}^j$ defines the density at time t of investors who distrust that market signal ($m \rightarrow 0$). The initial value $m_{t,k=0}^j$ can be exogenous or endogenous to the model setting. In particular, it can depend on F .

On this basis, at each time t , we assume that individual investors interact within each group j by subgroups of a given size for k sub-periods, in order to generate the group opinion for time t . In particular, for groups of size 3, the density after k successive updates is

$$m_{t,k}^j = (m_{t,k-1}^j)^3 + 3(m_{t,k-1}^j)^2(1 - m_{t,k-1}^j),$$

where $m_{t,k-1}^j$ is the proportion of agents who are confident in the market signal at a distance of $(k-1)$ updates from the initial time $t, k=0$. For groups of size 4, the density after n successive updates is

$$m_{t,k}^j = (m_{t,k-1}^j)^4 + 4(m_{t,k-1}^j)^3\{1 - m_{t,k-1}^j\} + 6(1-p)(m_{t,k-1}^j)^2\{1 - m_{t,k-1}^j\}^2,$$

where $m_{t,k-1}^j$ is the proportion of agents who are confident in the market signal at a distance of $(k-1)$ updates from the initial time $t, k=0$. The last term includes the tie case contribution (where two "believers" confronted with two "distrusters") weighted with the probability p . Then, the social mood (density) goes down to 0 with probability $(1-p)$ and up to 1 with probability p . For a mixture of group sizes with the probability distribution a_i with the constraint $\sum_{i=1}^L a_i = 1$ - where L is the largest group size and i refers to the group size:

$$m_{t,k}^j = \sum_{i=1}^L a_i \left\{ \sum_{j=N[\frac{i}{2}+1]}^i C_j^i (m_{t,k-1}^j)^j (1 - m_{t,k-1}^j)^{(i-j)} + (1-p)V(i)C_{\frac{i}{2}}^i (m_{t,k-1}^j)^{\frac{i}{2}} (1 - m_{t,k-1}^j)^{\frac{i}{2}} \right\}, \quad (2)$$

where $C_j^i \equiv \frac{i!}{(i-j)!j!}$, $N[\frac{i}{2}+1] \equiv \text{Integer Part of } (\frac{i}{2}+1)$, $m_{t,k-1}^j$ is the proportion of agents who believe in the market signal after $(k-1)$ updates, and $V(i) \equiv N[\frac{i}{2}] - N[\frac{i-1}{2}]$. This implies $V(i) = 1$ for i even and $V(i) = 0$ for i odd. The proportion of "distrusters" is then $1 - m_{t,k}^j$.

It is worth to emphasize that the Galam model of opinion dynamics tangles up three main mechanisms to produce a threshold opinion dynamics among two competing choices within an ensemble of investors. The first mechanism is exogenous and combines all effects which act directly and individually on the agent to influence its own personal choice, here to trust or distrust the current trend of the market. It determines the initial share $m_{t,k=0}^j$ of investors who are respectively confident to or distrusting the market price trend. The two other mechanisms are endogenous to the ensemble of interacting investors.

One mechanism embeds a social mimetic effect using a local majority rule, i.e., agents confront their actual choice with the ones of a small group of other agents and update their respective choices following the choice which was locally

majority within the group. At the collective global level, this interactive process produces a threshold dynamics for which the tipping point is located at precisely fifty percent: The choice which starts with an initial support of more than fifty percent of the investors will drive the market along its direction.

The second mechanism is more subtle and depends on the occurrence of a local doubt within a group of investors which are settling their respective opinions. In such a case, all the involved agents converge to just one common belief about the market price trend and adopt it as their own choice. Accordingly, within an ensemble of investors, with $m_{t,k}^j$ percent of them expecting the trend to be positive, a local doubting group of even size may decide to either trust the trend with a probability of p and distrust it with a probability of $(1 - p)$.

The breaking contribution of the leading common belief is to unbalance drastically the threshold dynamics by placing the tipping point at a value which can be as low as 15% for the choice which goes along the common belief, and as high as 85% for the choice which contradicts the common belief [15, 16]. For the case of group of size four used in this work, we have respectively 23% and 77% for the tipping points. This second mechanism illustrates how the common belief shared by some groups of investors can shape substantially the working of the market pricing [7] over time.

4 The formation of the market clearing price

The formation of the market clearing price p_{t+1}^* over time depends on the aggregation of individual bids of demand and supply at each period t . In particular, every shareholder ($j = S$) i wishes to sell if $p_{t+1}^* \geq E_t(p_{t+1})|_i^S$, while every potential buyer ($j = D$) i wishes to buy if $p_{t+1}^* \leq E_t(p_{t+1})|_i^D$. By assuming uniform distribution of individual investors within each group $j = S, D$, the individual price expectation $E_t(p_{t+1})|_i^j$ of investor i belonging to group j can be rewritten as a function of expectations expressed by investors $i = 0$ and $i = 1$ defined as follows:

$$\begin{aligned}\varepsilon_t|_0^j &\equiv \left(E_{t-1}(p_t)|_0^j - p_t \right) \\ \varepsilon_t|_1^j &\equiv \left(E_{t-1}(p_t)|_1^j - p_t \right).\end{aligned}$$

Individual price expectation by investor i may then be described as follows:

$$E_t(p_{t+1})|_i^j = p_t + m_{t,k}^j (p_t - p_{t-1}) - \left(\beta_0^j (1 - \varphi_i) \varepsilon_{0,t}^j + \beta_1^j \varphi_i \varepsilon_{1,t}^j \right) + \varphi_i \gamma^j F_t$$

Aggregated demand and supply are now defined by the focal prices of four representative agents with $i = 0$ and $i = 1 \forall j = S, D$. By defining:

$$\begin{aligned}\overline{P}_t^j &\equiv \max \arg \left[E_t(p_{t+1})|_{i=0}^j ; E_t(p_{t+1})|_{i=1}^j \right] \\ \underline{P}_t^j &\equiv \min \arg \left[E_t(p_{t+1})|_{i=0}^j ; E_t(p_{t+1})|_{i=1}^j \right],\end{aligned}$$

the aggregate functions of supply x_{t+1}^S and demand x_{t+1}^D integrate individual bids as follows:

$$\begin{cases} x_{t+1}^S = \int_{\underline{P}_t^S}^{p_{t+1}^*} \frac{1}{\overline{P}_t^S - \underline{P}_t^S} dx \\ x_{t+1}^D = \int_{p_{t+1}^*}^{\overline{P}_t^D} \frac{1}{\overline{P}_t^D - \underline{P}_t^D} dx \end{cases}$$

or, equivalently:

$$x_{t+1}^S = \begin{cases} 0 & \text{if } p_{t+1}^* \leq \underline{P}_t^S \\ \frac{p_{t+1}^* - \underline{P}_t^S}{\overline{P}_t^S - \underline{P}_t^S} & \text{if } \underline{P}_t^S < p_{t+1}^* < \overline{P}_t^S \\ 1 & \text{if } p_{t+1}^* \geq \overline{P}_t^S \end{cases} \quad (3)$$

$$x_{t+1}^D = \begin{cases} 1 & \text{if } p_{t+1}^* \leq \underline{P}_t^D \\ \frac{\overline{P}_t^D - p_{t+1}^*}{\overline{P}_t^D - \underline{P}_t^D} & \text{if } \underline{P}_t^D < p_{t+1}^* < \overline{P}_t^D \\ 0 & \text{if } p_{t+1}^* \geq \overline{P}_t^D. \end{cases} \quad (4)$$

The necessary condition for the existence of a market clearing price p_{t+1}^* (implying that both demand and supply are different from zero) is

$$\underline{P}_t^S \leq p_{t+1}^* \leq \overline{P}_t^D$$

This condition implies two different scenarios:

1. if $\overline{P}_t^D \leq \underline{P}_t^S$, there is not matching between demand and supply; therefore, no exchange transactions occur, and the share Exchange does not fix any updated clearing price at period t ; at the next period $t + 1$, investors will then observe a special no-clearing price p^{NC} generated by the market-making process according to some external rule or device;
2. if $\overline{P}_t^D > \underline{P}_t^S$, there is matching, and the market clearing price p^C is defined as the price that makes demand equal to supply.²

On this basis, the market clearing price at period t is

$$p_{t+1}^* = \begin{cases} p^{NC} & \text{if } \overline{P}_t^D \leq \underline{P}_t^S \\ p^C & \text{if } \overline{P}_t^D > \underline{P}_t^S \end{cases}$$

²The Walrasian auction is included by this scenario when the whole share offer is satisfied.

Let assume that the no-clearing price p^{NC} is fixed according to the following rule:

$$p^{NC} = p_t + \epsilon$$

where ϵ is the smallest tick value available on the share Exchange. Furthermore, concerning the clearing price p^C , demand is equal to supply if

$$\frac{p^C - \underline{P}_t^S}{\overline{P}_t^S - \underline{P}_t^S} = \frac{\overline{P}_t^D - p^C}{\overline{P}_t^D - \underline{P}_t^D}, \text{ implying that}$$

$$p^C = \frac{\overline{P}_t^D (\overline{P}_t^S - \underline{P}_t^S) + \underline{P}_t^S (\overline{P}_t^D - \underline{P}_t^D)}{(\overline{P}_t^D - \underline{P}_t^D) + (\overline{P}_t^S - \underline{P}_t^S)}$$

Therefore, the market clearing price p_{t+1}^* at time t is:

$$p_{t+1}^* = \begin{cases} p^{NC} = p_t + \epsilon & \text{if } \overline{P}_t^D \leq \underline{P}_t^S \\ p^C = \frac{\overline{P}_t^D (\overline{P}_t^S - \underline{P}_t^S) + \underline{P}_t^S (\overline{P}_t^D - \underline{P}_t^D)}{(\overline{P}_t^D - \underline{P}_t^D) + (\overline{P}_t^S - \underline{P}_t^S)} & \text{if } \overline{P}_t^D > \underline{P}_t^S. \end{cases} \quad (5)$$

5 The dynamics of the market clearing price

In order to analyze the dynamics of the market clearing price (when it exists, i.e., $p_{t+1}^* = p^C$) over time, let define $\forall j = S, D$:

$$\mathbf{P}^j(n) \equiv \sum_{n=1}^t \left(-\beta_0^j \right)^n \left(p_{t-n} + m_{t-n}^j (p_{t-n} - p_{t-n-1}) - p_{t-n+1} \right)$$

$$\mathbf{F}^j(n) \equiv \sum_{n=0}^t \left[\left(-\beta_0^j \right)^n (\gamma^j F_{t-n}) \right]$$

$$\mathbf{L}^j(\mathbf{P}(n), \mathbf{F}(n)) \equiv \left[\frac{\left| (\beta_1^j - \beta_0^j) \cdot \mathbf{P}^j(n) + \mathbf{F}^j(n) \right|}{\sum_{j=S,D} \left| (\beta_1^j - \beta_0^j) \cdot \mathbf{P}^j(n) + \mathbf{F}^j(n) \right|} \right]^{-1}$$

$$\mathbf{M}^j(\mathbf{P}(n), \mathbf{F}(n)) \equiv (\beta_1^j - \beta_0^j) \cdot \mathbf{P}^j(n) + \mathbf{F}^j(n).$$

Accordingly,

$$E_t(p_{t+1})|_i^j = p_t + m_{t,k}^j (p_t - p_{t-1}) + \beta_0^j \cdot \mathbf{P}^j(n) + \varphi_i \cdot \mathbf{M}^j(\cdot).$$

The four representative agents are then described as follows:

$$\begin{aligned} \forall j = S, D \text{ with } \varphi_i = 0: E_t(p_{t+1})|_{i=0}^j &= p_t + m_{t,k}^j (p_t - p_{t-1}) + \beta_0^j \cdot \mathbf{P}^j(n) \\ \forall j = S, D \text{ with } \varphi_i = 1: E_t(p_{t+1})|_{i=0}^j &= p_t + m_{t,k}^j (p_t - p_{t-1}) + \beta_1^j \cdot \mathbf{P}^j(n) + \mathbf{F}^j(n). \end{aligned}$$

By computation, the market clearing price function can be rewritten as follows:

$$p_{t+1}^* = p_t + \sum_{j=S,D} \left[\frac{\left(m_{t,k}^j (p_t - p_{t-1}) + \mathbf{P}^j(\cdot) \right)}{\mathbf{L}^{-j}(\cdot)} \right] + \begin{cases} \left[\frac{\mathbf{M}^D(\cdot)}{\mathbf{L}^S(\cdot)} \right] & \text{if } \mathbf{M}^j(\cdot) > 0 \ \forall j \\ \left[\frac{\mathbf{M}^S(\cdot)}{\mathbf{L}^D(\cdot)} \right] & \text{if } \mathbf{M}^j(\cdot) < 0 \ \forall j \\ \sum_{j=S,D} \left[\frac{\mathbf{M}^j(\cdot)}{\mathbf{L}^{-j}(\cdot)} \right] & \text{if } \mathbf{M}^D(\cdot) > 0 \text{ and } \mathbf{M}^S(\cdot) < 0 \\ 0 & \text{if } \mathbf{M}^D(\cdot) < 0 \text{ and } \mathbf{M}^S(\cdot) > 0 \end{cases} \quad (6)$$

Accordingly, the pattern of market clearing price p_{t+1}^* is based on the historical price p_t by adding two further elements. The first element comprises (for both $j = S$ and $j = D$) two sub-elements:

- the numerator, $m_{t,k}^j (p_t - p_{t-1}) + \mathbf{P}^j(n)$, is independent from signal F_t and dependent on the price trend $\Delta_t(p^*)$ weighted by the current group mood $m_{t,k}^j$,³ and its weighted past series $\mathbf{P}^j(n)$;
- the denominator, $\mathbf{L}^j(\cdot)$, depends on both $\mathbf{F}^j(n)$ which represents the weighted fundamental signal trend series, and $\mathbf{P}^j(n)$ which represents the weighted market price trend series; for each group j , this sub-element weights the contribution of the price trend series to the formation of the market clearing price at time t .

The second element depends on both weighted past series $\mathbf{F}^j(\cdot)$ and $\mathbf{P}^j(\cdot)$. In particular, if $\mathbf{M}^j(\cdot)$ is positive (negative) for both groups, then this element increases (decreases) the market clearing price. Moreover, if $\mathbf{M}^j(\cdot)$ is negative for shareholders ($j = S$) while it is positive for potential investors ($j = D$), then the divergence between groups is mutually balanced on the marketplace. On the contrary, if $\mathbf{M}^j(\cdot)$ is positive for shareholders while negative for potential investors, then the divergence makes the whole element equal to zero.

³Remember that $m_{t,k}^j \rightarrow 1$ implies full weight to this information in order to build individual price expectations. The mood m can be influenced by the fundamental signal F , and is the final result of the dynamic interaction within the group j for k steps occurring between $t-1$ and t .

In sum, the formation of share prices over time depends respectively on the dynamics of the fundamental signal F from one side, and the dynamics of the clearing market price p from another side. Both dynamics are shaped by the ongoing evolution of individual and group opinions (and related bids) captured by the structure of the model.

6 An illustrative analysis of a particular specification of the model

This section shall illustrate the theoretical contribution of our model by visualizing some particular cases. For this purpose, let assume that $\beta_i^j = \beta$ and $\gamma^j = \gamma \forall j = S, D$ and $\forall i$. This specification implies that group heterogeneity is captured by the group mood $m_{t,k}^j$ and leads to the following proposition:

Proposition 1 *If $\beta_i^j = \beta$ and $\gamma^j = \gamma \forall j = S, D$ and $\forall i$, the group mood $m_{t,k}^j$ subsumes all the group heterogeneities between demand and supply; then, there exists only one market clearing price case instead of the four cases defined above.*

In particular, the individual price expectation function becomes:

$$\begin{aligned} E_t(p_{t+1})|_i^j &= p_t + m_{t,k}^j (p_t - p_{t-1}) + \\ &\quad \sum_{n=1}^t \left[(-\beta)^n \left(p_{t-n} - m_{t-n}^j (p_{t-n} - p_{t-n-1}) - p_{t-n+1} \right) \right] + \\ &\quad \varphi_i \sum_{n=0}^t [(-\beta)^n (\gamma F_{t-n})] \\ \text{or } E_t(p_{t+1})|_i^j &= p_t + m_{t,k}^j (p_t - p_{t-1}) + \mathbf{P}^j(n) + \varphi_i \gamma^j \mathbf{F}^j(n). \end{aligned}$$

Concerning the formation of the market clearing price, for $\beta_i^j = \beta \forall i, j$, $\mathbf{L}^j(\cdot) = 2$. Therefore, closer are $\beta_i^j \forall i, j$, closer is $\mathbf{L}^j(\cdot)$ to 2, implying that the whole first element of equation 6 tends to become independent from the fundamental signal series $\mathbf{F}^j(n)$. Furthermore, when $\beta_1^j, \beta_0^j = \beta^j$, $\mathbf{M}^j(\cdot) = \mathbf{F}^j(n) \forall j$: Closer are β_0^j and $\beta_1^j \forall j$, closer is $\mathbf{M}^j(\cdot)$ to $\mathbf{F}^j(n)$ that is independent from the market price trend series $\mathbf{P}^j(n)$. Therefore, this specification clearly distinguishes the dual structure of the market clearing price dynamics which is driven by two distinct factors: the market signal or trend $\Delta_t(p^*)$ weighted by the evolution of groups' market sentiments, and the fundamental signal F . The market clearing price becomes:

$$p_{t+1}^* = p_t + \frac{1}{2} \sum_{j=S,D} \left(m_{t,k}^j (p_t - p_{t-1}) + \mathbf{P}^j(n) \right) + \left[\frac{\mathbf{F}(n)}{2} \right]$$

where

$$\begin{aligned}\mathbf{P}^j(n) &\equiv \sum_{n=1}^t (-\beta)^n \left(p_{t-n} + m_{t-n}^j (p_{t-n} - p_{t-n-1}) - p_{t-n+1} \right) \\ \mathbf{F}(n) &\equiv \sum_{n=0}^t [(-\beta)^n (\gamma F_{t-n})]\end{aligned}$$

Accordingly, the dynamics of the market clearing price (when it exists) is denoted as follows:

$$\Delta_{t+1}(p^*) \equiv p_{t+1} - p_t = f\left(\Delta_t(p^*), m_{t,k}^j\right) + g(\mathbf{F}(n))$$

This price pattern comprises two different elements. The first element, $f\left(\Delta_t(p^*), m_{t,k}^j\right)$, is a group factor that depends on the market signal $\Delta_t(p^*)$ weighted by the group mood $m_{t,k}^j$ that is collectively assigned to the market price trend by group j at time t . The second element, $g(\mathbf{F}(n)) = \frac{\mathbf{F}(n)}{2}$, depends on the weighted trend of the fundamental signal F_t , with $\mathbf{F}^j(n) = \mathbf{F}^D(n) = \mathbf{F}^S(n)$ in this particular specification. Consequently, if $\mathbf{F}(n)$ is positive (negative), then $g(\mathbf{F}(n))$ proportionally increases (decreases) the market clearing price at time t .

6.1 Illustrative case of a constant trend in the fundamental signal

Let illustrate this particular specification of the model when the fundamental signal experiences an alternate positive and negative trend: $F_t = \pm 0.1$ every 10 periods of t . Let assume: $\beta_i^j = \beta = 0.5$; $\gamma^j = \gamma = 1$; $\epsilon = 0.01$; $p_0 = 10$; $F_0 = 0$; $m_{t,k=0}^S = 0.6$; $m_{t,k=0}^D = 0.4$; $\varepsilon_{t=0|i}^j = 0.1 \cdot (U(0;1) - U(0;1))$, $\forall j = S, D$ and $\forall i$. The group interaction is based on groups of size 3, 4 (with $p^D = p^S = 0.3$) and 4 (with $p^D = 0.3$ and $p^S = 0.3$). The probability p means here that, in case of group inderterminacy, the group belief tends to trust the market ($p \rightarrow 1$) or not ($p \rightarrow 0$). We perform simulations for periods from $t = 1$ to $t = 100$, with various k from 0 to 7. In this case, the market clearing price and the fundamental signal change at the same rhythm t , while the market sentiment changes at its rhythm k from 0 (no steps, implying no change from the initial value) to 7 (seven steps between $t - 1$ and t). Figures 1a,b,c illustrate the result.

The simulation shows that changes in the market sentiment exacerbates the impact of the fundamental signal on the market clearing price over time. The resulting market price remains under- or over-valuated the shares relative to their fundamental price computed as follows:⁴

$$p_t^F = p_{t-1}^F + F_{t-1} = p_0 + \sum_{n=0}^{t-1} F_n.$$

⁴By definition, the first market price and the first fundamental price are equal.

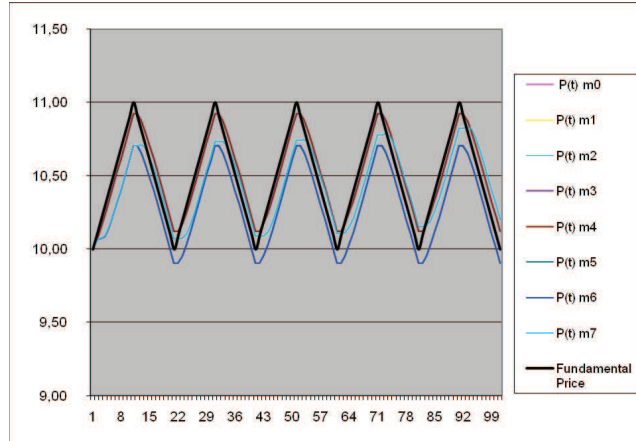


Figure 1: Simulation for groups of size 3

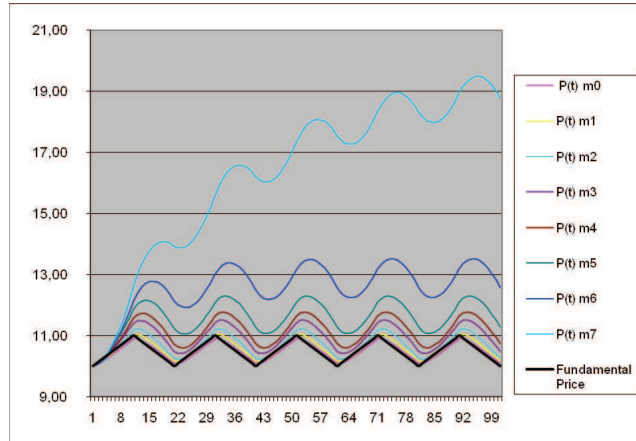


Figure 2: Simulation for groups of size 4 ($p^D = p^S = 0.3$)

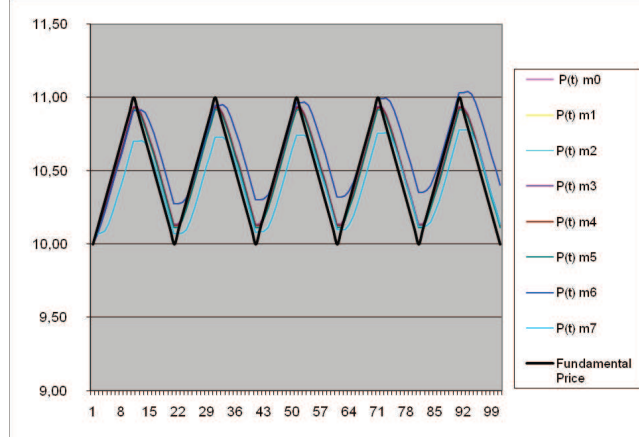


Figure 3: Simulation for groups of size 4 (with $p^D = 0.6$ and $p^S = 0.4$)

6.2 Illustrative case of market exuberance

Under the same conditions, let assume that the fundamental signal experiences a random pattern: $F_t = U(0;1) - U(0;1) \forall t \geq 1$. We perform simulations for periods from $t = 1$ to $t = 100$, with various k from 0 to 7. In this case, the market clearing price and the fundamental signal change at the same rhythm t , while the market sentiment changes at its rhythm k from 0 (no change) to 7 (seven steps between $t - 1$ and t). Figures 2a,b,c illustrate the result.

The simulation shows that changes in the market sentiment exacerbates the market exuberance around the path provided by the fundamental price p_t^F . This result is in line with theoretical and empirical analyses of market exuberance discussed by [27] and [28] among others.

6.3 Illustrative case of market disconnection from fundamental price

The latter case shows a distinctive pattern where the market price disconnects from the fundamental price over time. Figures 3a,b,c illustrate this result.

The simulation shows that the dynamics of the clearing price may be disconnected by the dynamics of fundamental price p_t^F over time. This implies that the formation of a market clearing price over time is not sufficient to assure that market pricing is aligned on the fundamental price that arises from fundamentals that are common knowledge among heterogeneous market participants.

7 Conclusive remarks

Financial economic models often assume that investors know (or agree on) the fundamental value of the firm's shares that are traded, easing the passage from

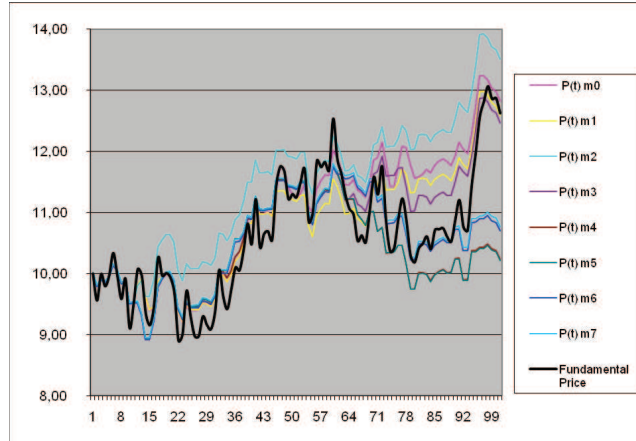


Figure 4: Simulation for groups of size 3

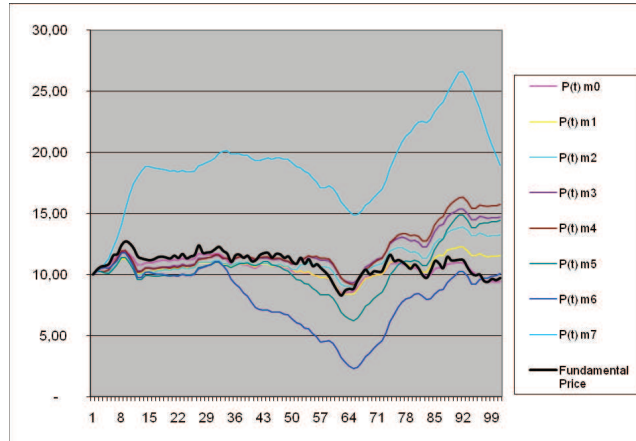


Figure 5: Simulation for groups of size 4 ($p^D = p^S = 0.3$)

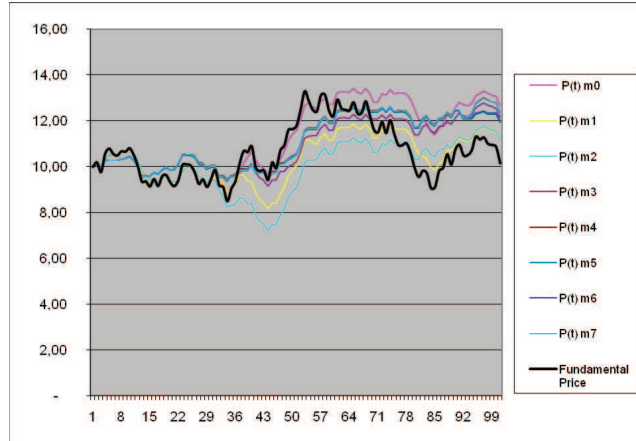


Figure 6: Simulation for groups of size 4 (with $p^D = 0.6$ and $p^S = 0.4$)

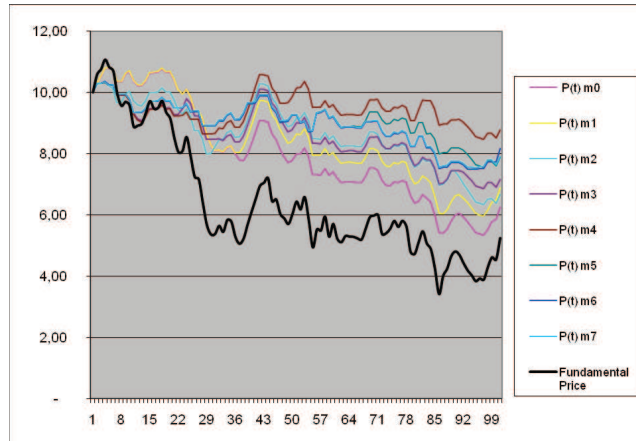


Figure 7: Simulation for groups of size 3

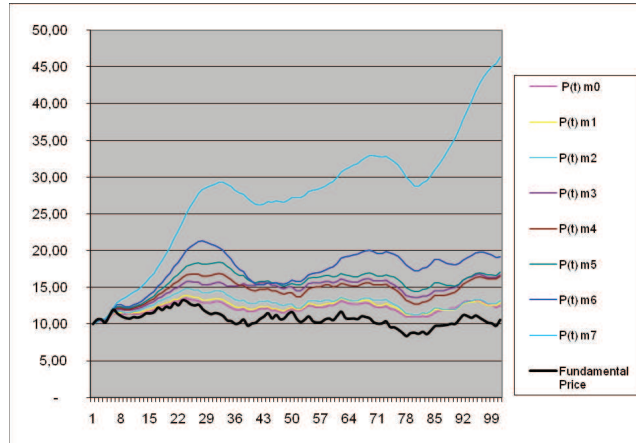


Figure 8: Simulation for groups of size 4 ($p^D = p^S = 0.3$)

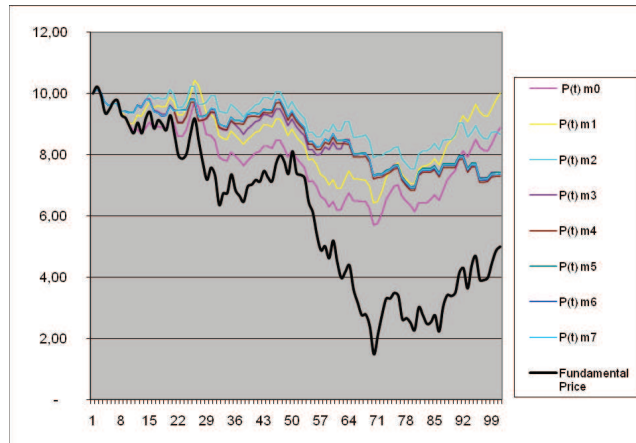


Figure 9: Simulation for groups of size 4 (with $p^D = 0.6$ and $p^S = 0.4$)

the individual to the collective dimension of the financial system generated by the Share Exchange over time. Our model relaxes that heroic assumption of one unique "true value" and deals with the formation of share market prices through the dynamic formation of individual and social opinions (or beliefs) based upon a fundamental signal of economic performance and position of the firm, the forecast revision by heterogeneous individual investors, and their social mood or sentiment about the ongoing state of the market pricing process. Market clearing price formation is then featured by individual and group dynamics that make its collective dimension irreducible to its individual level.

This dynamic holistic approach provides a better understanding of the market exuberance generated by the Share Exchange over time. This exuberance depends not only on individual biases or mistakes, but also on dynamic and collective dimensions that arise from the interaction of individuals among them and with evolving collective structures over time. Our model captures this collective dimension through the evolution of available common knowledge on the economic performance and position of the firm (fundamental or firm-specific information), as well as through the evolving social mood or sentiment on the current state of the market (or the industry, or the whole economy). While the former can be related to information release by accounting reporting and disclosure, the latter can be related to investors' confidence and financial analysts' consensus, and their respective evolution over socio-economic time and space where the financial market is embedded.

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Appendix

For sake of completeness, we provide here various other versions of the market price equation. Starting from equation 6, let define:

$$\begin{aligned}\lambda_t &\equiv \frac{|\mathbf{M}^S|}{|\mathbf{M}^S| + |\mathbf{M}^D|} = \mathbf{L}^S(\mathbf{P}(n), \mathbf{F}(n))^{-1} \\ (1 - \lambda_t) &\equiv \frac{|\mathbf{M}^D|}{|\mathbf{M}^S| + |\mathbf{M}^D|} = \mathbf{L}^D(\mathbf{P}(n), \mathbf{F}(n))^{-1} \\ \text{with } \mathbf{M}^j(\cdot) &= (\beta_1^j - \beta_0^j) \cdot \mathbf{P}^j(n) + \mathbf{F}^j(n) = -(\beta_1^j \varepsilon_{1,t}^j - \beta_0^j \varepsilon_{0,t}^j) + \gamma^j F_t\end{aligned}$$

Or, equivalently:

$$\begin{aligned}\lambda_t &= \frac{\left| \begin{aligned} &(\beta_1^S - \beta_0^S) \sum_{n=1}^t [(-\beta_0^S)^n (p_{t-n} + m_{t-n}^S (p_{t-n} - p_{t-n-1}) - p_{t-n+1})] + \\ &\sum_{n=0}^t [(-\beta_0^D)^n (\gamma^D F_{t-n})] \end{aligned} \right|}{\sum_{j=S,D} \left| \begin{aligned} &(\beta_1^j - \beta_0^j) \sum_{n=1}^t [(-\beta_0^j)^n (p_{t-n} + m_{t-n}^j (p_{t-n} - p_{t-n-1}) - p_{t-n+1})] + \\ &\sum_{n=0}^t [(-\beta_0^j)^n (\gamma^j F_{t-n})] \end{aligned} \right|} \\ \text{and} \\ (1 - \lambda_t) &= \frac{\left| \begin{aligned} &(\beta_1^D - \beta_0^D) \sum_{n=1}^t [(-\beta_0^D)^n (p_{t-n} + m_{t-n}^D (p_{t-n} - p_{t-n-1}) - p_{t-n+1})] + \\ &\sum_{n=0}^t [(-\beta_0^D)^n (\gamma^D F_{t-n})] \end{aligned} \right|}{\sum_{j=S,D} \left| \begin{aligned} &(\beta_1^j - \beta_0^j) \sum_{n=1}^t [(-\beta_0^j)^n (p_{t-n} + m_{t-n}^j (p_{t-n} - p_{t-n-1}) - p_{t-n+1})] \\ &+ \sum_{n=0}^t [(-\beta_0^j)^n (\gamma^j F_{t-n})] \end{aligned} \right|}\end{aligned}$$

Therefore, the market clearing equation can be rewritten as follows:

$$\begin{aligned}
p_{t+1}^* &= p_t + \lambda_t m_t^D (p_t - p_{t-1}) + (1 - \lambda_t) m_t^S (p_t - p_{t-1}) + \\
&\quad \lambda_t (\beta_0^D \varepsilon_{0,t}^D) + (1 - \lambda_t) (\beta_0^S \varepsilon_{0,t}^S) + \\
&\quad \left\{ \begin{array}{ll} \lambda_t [(\beta_1^D \varepsilon_{1,t}^D - \beta_0^D \varepsilon_{0,t}^D) + \gamma_t^D F_t] & \text{if } \mathbf{M}^j > 0 \\ (1 - \lambda_t) [(\beta_1^S \varepsilon_{1,t}^S - \beta_0^S \varepsilon_{0,t}^S) + \gamma_t^S F_t] & \text{if } \mathbf{M}^j < 0 \\ \lambda_t [(\beta_1^D \varepsilon_{1,t}^D - \beta_0^D \varepsilon_{0,t}^D) + \gamma_t^D F_t] + & \text{if } \mathbf{M}^D > 0 \\ (1 - \lambda_t) [(\beta_1^S \varepsilon_{1,t}^S - \beta_0^S \varepsilon_{0,t}^S) + \gamma_t^S F_t] & \text{and } \mathbf{M}^S < 0 \\ 0 & \text{if } \mathbf{M}^D < 0 \\ & \text{and } \mathbf{M}^S > 0 \end{array} \right.
\end{aligned}$$

Or, equivalently:

$$\begin{aligned}
p_{t+1}^* &= p_t + \lambda_t m_t^D (p_t - p_{t-1}) + (1 - \lambda_t) m_t^S (p_t - p_{t-1}) + \\
&\quad \lambda_t \sum_{n=1}^t \left[(-\beta_0^D)^n (p_{t-n} + m_{t-n}^D (p_{t-n} - p_{t-n-1}) - p_{t-n+1}) \right] + \\
&\quad (1 - \lambda_t) \sum_{n=1}^t \left[(-\beta_0^S)^n (p_{t-n} + m_{t-n}^S (p_{t-n} - p_{t-n-1}) - p_{t-n+1}) \right] + \\
&\quad \left\{ \begin{array}{ll} \lambda_t [\mathbf{M}^D(\cdot)] & \text{if } \mathbf{M}^j > 0 \ \forall j \\ (1 - \lambda_t) [\mathbf{M}^S(\cdot)] & \text{if } \mathbf{M}^j < 0 \ \forall j \\ \lambda_t [\mathbf{M}^D(\cdot)] + (1 - \lambda_t) [\mathbf{M}^S(\cdot)] & \text{if } \mathbf{M}^D > 0 \\ & \text{and } \mathbf{M}^S < 0 \\ 0 & \text{if } \mathbf{M}^D < 0 \\ & \text{and } \mathbf{M}^S > 0 \end{array} \right.
\end{aligned}$$